QUASI-ACTIONS AND ROUGH CAYLEY GRAPHS FOR LOCALLY COMPACT GROUPS

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Abstract. We define the notion of rough Cayley graph for compactly generated locally compact groups in terms of quasi-actions. We construct such a graph for any compactly generated locally compact group and show uniqueness up to quasi-isometry. A class of examples is given by the Cayley graphs of cocompact lattices in compactly generated groups. As an application, we show that a compactly generated group has polynomial growth if and only if its rough Cayley graph has polynomial growth (same for intermediate and exponential growth). Moreover, a unimodular compactly generated group is amenable if and only if its rough Cayley graph is amenable as a metric space.

1. Introduction

The purpose of this paper is to introduce the notion of rough Cayley graph for compactly generated locally compact groups and show its usefulness in abstract harmonic analysis. Another central notion of the paper is quasi-action of a locally compact group on a metric space. Similar concepts have been fruitfully employed in the study of geometric group theory in the discrete case (see for example [4, 8]), and there is no doubt that the corresponding concepts will be useful in the study of locally compact groups. Indeed, Krön and Möller [6] have already studied the case of totally disconnected groups, introducing the notion of rough Cayley graph in that context. Our definition goes beyond the totally disconnected case where the two definitions coincide.

One can study coarse geometry of compactly generated locally compact groups using the word metric with respect to a compact generating set, as done for example in [3, 1]. The rough Cayley graph forgets the local information but retains the coarse information, thereby giving an alternative (but equivalent) approach to coarse geometry of locally compact groups.

As an application of rough Cayley graphs, we shall show that the growth of a compactly generated group can be described in terms of its rough Cayley graph. Moreover, we shall show that when the compactly

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generated group is unimodular, it is amenable if and only if its rough Cayley graph is amenable in the sense of [2]. To give examples of rough Cayley graphs, we shall show that the Cayley graph of a cocompact lattice in a compactly generated group is the rough Cayley graph of the ambient group.

2. Quasi-actions

We start with some terminology from coarse geometry. When X is a metric space and $A \subseteq X$, we write

$$N_r(A) = \{\, x \in X;\, d(x,a) \leq r \text{ for some } a \in A \,\}$$

(and $N_r(x) := N_r(\{x\})$ for $x \in X$). Let $C \geq 1$ and $r \geq 0$ be constants. A map $f\colon X \to Y$ between metric spaces is (C,r)-*quasi-isometric embedding* if

$$C^{-1} d_X(x,y) - r \leq d_Y(f(x), f(y)) \leq C d_X(x,y) + r$$

for every $x, y \in X$. In this case, f is a (C,r)-*quasi-isometry* if f is also r-*coarsely onto*: $Y = N_r(f(X))$.

A (C,r)-*quasi-action* of a locally compact group G on a metric space X is defined by a collection of maps $x \mapsto s \cdot x\colon X \to X$ satisfying the following conditions:

(i) $x \mapsto s \cdot x\colon X \to X$ is a (C,r)-quasi-isometry for every $s \in G$
(ii) $e \cdot x = x$ for every $x \in X$ (we denote the identity of G by e)
(iii) $d(s \cdot (t \cdot x), (st) \cdot x) \leq r$ for every $s, t \in G$ and $x \in X$
(iv) $K \cdot x$ is bounded whenever $K \subseteq G$ is compact and $x \in X$.

A quasi-action is *cobounded* if there is $r \geq 0$ such that for every $x \in X$ the map $s \mapsto s \cdot x\colon G \to X$ is r-coarsely onto. In other words, a cobounded quasi-action is coarsely transitive. To simplify the notation, we shall use the same 'r' to denote the constant associated with a (C,r)-quasi-action as well as the constant associated with coboundeness.

We say that a quasi-action is *proper* if for every $R \geq 0$ and for every $x \in X$ the set

$$\{\, s \in G;\, d(s \cdot x, x) \leq R \,\}$$

is relatively compact (i.e. its closure is compact).

'Compactly generated group' refers to a compactly generated *locally compact* group. We consider such groups as metric spaces equipped with the word metric with respect to a generating set that is a compact symmetric neighbourhood of the identity. Note that all such metric spaces are quasi-isometric because every pair of compact generating neighbourhoods are contained to some powers of each other.

The following result is the analogue of the Švarc–Milnor lemma for quasi-actions of locally compact groups (see [4, 10] for the classical Švarc–Milnor lemma). We shall need this result to show the uniqueness of rough Cayley graphs in the following section.

A metric space is *geodesic* if every two points can be joined by a geodesic path.

Theorem 1. *Suppose that G is a locally compact group that has a cobounded, proper quasi-action $(s, x) \mapsto s \cdot x \colon G \times X \to X$ on a geodesic metric space X. Then G is compactly generated and for a fixed $x \in X$ the map $s \mapsto s \cdot x \colon G \to X$ is a quasi-isometry.*

Proof. Let $C \geq 1$ and $r \geq 0$ be the constants associated with the quasi-action. Fix $x \in X$. By coboundedness, $X = N_r(G \cdot x)$. Choose $R = C(2r+1) + 3r$ and let K be a compact symmetric neighbourhood of the identity containing

$$\{\, s \in G;\, d(s \cdot x, x) \leq R \,\}$$

(K exists because the quasi-action is proper). We shall first show that K generates G.

Let $u \in G \setminus K$ and let $n \geq 2$ be the unique integer such that

$$R - r + n - 2 \leq d(u \cdot x, x) < R - r + n - 1.$$

Since X is geodesic, there is a path x, y_1, y_2, …, $y_n = u \cdot x$ such that $d(x, y_1) \leq R - r$ and $d(y_i, y_{i+1}) \leq 1$ for every $i = 1, 2, \ldots, n-1$. Put $s_n = u$ and for each $i = 1, \ldots, n-1$ choose $s_i \in G$ such that $y_i \in N_r(s_i \cdot x)$. Then $s_1 \in K$ and for every $i = 1, \ldots, n-1$

$$d(s_i^{-1} s_{i+1} \cdot x, x) \leq C d(s_{i+1} \cdot x, s_i \cdot x) + 3r \leq C(2r+1) + 3r = R$$

and so $s_i^{-1} s_{i+1} \in K$. Therefore

$$u = s_n = s_1 (s_1^{-1} s_2)(s_2^{-1} s_3) \ldots (s_{n-1}^{-1} s_n) \in K^n.$$

It follows that K generates G, so we may use the word metric on G with respect to K. Another consequence of the preceding calculation is that for every $s, t \in G$ with $s^{-1}t \notin K$ there is $n \geq 2$ such that $s^{-1}t \in K^n$ and

$$n \leq d(s^{-1}t \cdot x, x) + 2 + r - R.$$

Hence

$$d_G(s,t) \leq C d(s \cdot x, t \cdot x) + 3r + 2 + r - R.$$

Including the case when $s^{-1}t \in K$, we have

$$d_G(s,t) \leq C d(s \cdot x, t \cdot x) + 1$$

for every $s, t \in G$.

Let

$$M = \sup\{\, d(s \cdot x, x);\, s \in K \,\},$$

which exists because K is compact. For every s_1, s_2, …, s_n in K, we have

$$\begin{aligned} d(s_1 \ldots s_n \cdot x, x) &\leq d(s_1 \cdot x, x) + d(s_1 s_2 \cdot x, s_1 \cdot x) + \cdots \\ &\qquad + d(s_1 \ldots s_n \cdot x, s_1 \ldots s_{n-1} \cdot x) \\ &\leq M + (n-1)(CM + 2r) \end{aligned}$$

Suppose that $d_G(s,t) = n$ so that $s^{-1}t \in K^n$. Then

$$d(s \cdot x, t \cdot x) \leq Cd(s^{-1}t \cdot x, x) + 3Cr \leq C(CM + 2r)d_G(s,t) + 3Cr$$

by the preceding calculation.

Coboundedness of the quasi-action means that the map $s \mapsto s \cdot x$ is coarsely onto, so it is a quasi-isometry. □

3. Rough Cayley graph

Throughout this section, G denotes a compactly generated locally compact group. We consider graphs as discrete metric spaces consisting of vertices and equipped with the graph metric. A graph is said to be *uniformly locally finite* is there is a uniform bound on the degree of vertices.

Of course it is too much to ask that a quasi-action of a locally compact group G on a graph X is continuous, but a quasi-action should respect the local structure of G on some level. A set $T \subseteq G$ is *right uniformly discrete* with respect to a relatively compact neighbourhood V of the identity if $tV \cap t'V = \emptyset$ whenever $t \neq t'$ are in T. A right uniformly discrete subset is maximal if it is not properly contained to another subset that is right uniformly discrete with respect to V (this is equivalent to $TVV^{-1} = G$). *Left uniformly discrete* subset is defined analogously, moving V to the left-hand side of t and t'. We say that a quasi-action of G on X is *uniformly represented* if for every $x \in X$ there is a right uniformly discrete subset $T \subseteq G$ such that

$$G \cdot x = T \cdot x.$$

A *rough Cayley graph* of G is a uniformly locally finite, connected graph X such that G has a proper, cobounded, uniformly represented quasi-action on X.

Theorem 2. *There exists a rough Cayley graph for any compactly generated locally compact group G.*

Proof. The compact case is trivial so suppose that G is not compact.

Fix a compact symmetric neighbourhood U of the identity such that U generates G. We construct $X \subseteq G$ inductively, starting with $x_0 = e$. Suppose that we have picked x_0, x_1, …, x_n and let m be the smallest integer such that

$$U^m \not\subseteq \bigcup_{i=0}^{n} x_i U^2$$

(m exists because G is not compact). Pick

$$x_{n+1} \in U^m \setminus \bigcup_{i=0}^{n} x_i U^2.$$

Since $U^{m-1} \subseteq \bigcup_{i=0}^{n} x_i U^2$ we have that

$$x_{n+1} \in \bigcup_{i=0}^{n} x_i U^3. \tag{1}$$

Continuing the process, we get a set $X = \{x_i\}_{i=0}^{\infty}$. Since each U^m is compact, there exists k such that

$$U^m \subseteq \bigcup_{i=0}^{k} x_i U^2 :$$

otherwise U^m contains an infinite right uniformly discrete set by construction. Since U generates G, it follows that $G = \bigcup_{i=0}^{\infty} x_i U^2$. Therefore X is a maximal right uniformly discrete set with respect to U.

Define a graph structure on X by adjoining vertices $x, y \in X$ whenever $x^{-1}y \in U^3$. By (1), x_{n+1} is connected to one of the earlier x_i's and hence the graph X is connected.

Next we show that the graph is uniformly locally finite. Fix $x \in X$. Since

$$Ux^{-1}y \cap Uz^{-1}y \neq \emptyset \implies xU \cap zU \neq \emptyset,$$

the set $\{x^{-1}y\}_{y \in X}$ is left uniformly discrete with respect to U. Now U^3 may be covered by a finite number, say M, of right translates of U. No two points of the form $x^{-1}y$ can be in the same right translate of U, so at most M points of the form $x^{-1}y$ are in U^3. That is, there are at most M vertices y adjacent to x. The number M does not depend on x, so it is a uniform bound for the degrees of vertices.

Finally, we define a quasi-action of G on X that it is cobounded, proper and uniformly represented. Given $x \in X$, choose a symmetric neighbourhood V of the identity such that $x^{-1}V^2x \subseteq U$. Let $Y \subseteq G$ be a right uniformly discrete set with respect to V such that $G = YV^2$. Define first $s \cdot x$ for $s \in Y$ by choosing $s \cdot x \in sxU^2 \cap X$ (recall that $G = XU^2$ so this is possible). We may take $e \in Y$ and define $x \mapsto e \cdot x$ to be the identity map. Then put $sv \cdot x = s \cdot x$ for every $v \in V$, which is fine because Y is right uniformly discrete with respect to V. Finally, for $t \in G \setminus YV$, there is $s \in Y$ such that $t = su$ where $u \in V^2$. Put $t \cdot x = s \cdot x$.

In all cases, $t \cdot x = s \cdot x$ for some $s \in Y$ such that $t = su$ where $u \in V^2$. Since

$$s \cdot x \in tu^{-1}xU^2 \subseteq tx(x^{-1}V^2x)U^2 \subseteq txU^3,$$

we have

$$t \cdot x \in txU^3. \tag{2}$$

Moreover, the would-be quasi-action is uniformly represented because $G \cdot x = Y \cdot x$.

If $K \subseteq G$ is compact, then $Kx \subseteq U^m$ for some integer m (because U generates G and is a neighbourhood of the identity). Hence $K \cdot x \subseteq$

U^{m+3} by (2), which shows that $K \cdot x$ is bounded in the graph metric of X.

Fix s in G and let $x, y \in X$. Suppose that $d(x,y) = n$ so that $x^{-1}y \in U^{3n}$. By (2), $s \cdot x \in sxU^3$ and $s \cdot y \in syU^3$, so

$$(s \cdot x)^{-1}(s \cdot y) \in U^3 x^{-1} y U^3 \subseteq U^{3n+6}.$$

Hence

$$d(s \cdot x, s \cdot y) \leq d(x,y) + 2.$$

Similarly,

$$d(x,y) - 2 \leq d(s \cdot x, s \cdot y),$$

so $x \mapsto s \cdot x$ is a $(1,2)$-quasi-isometric embedding.

As for the coarse associativity condition, suppose that $s, t \in G$ and $x \in X$. Then $s \cdot (t \cdot x) \in s(t \cdot x)U^3$, $t \cdot x \in txU^3$ and $(st) \cdot x \in stxU^3$. It follows that $((st) \cdot x)^{-1}(s \cdot (t \cdot x)) \in U^9$ and so

$$d\big((st) \cdot x, s \cdot (t \cdot x)\big) \leq 3.$$

In particular,

$$d\big(x, s \cdot (s^{-1} \cdot x)\big) \leq 3,$$

which shows that $x \mapsto s \cdot x$ is a $(1,3)$-quasi-isometry.

The quasi-action is cobounded because $d(y, (yx^{-1}) \cdot x) \leq 1$ for every $x, y \in X$. It remains to show that the quasi-action is proper. Fix $R > 0$ and $x \in X$. We may take R to be an integer. If $d(s \cdot x, x) \leq R$, then $x^{-1}(s \cdot x) \in U^{3R}$. It follows from (2) that $s \in xU^{3R+3}x^{-1}$. The latter set is compact so we are done. □

In the next section, we shall see that a cocompact lattice in G determines the rough Cayley graph of G (if such lattice exists). This example covers the usual Cayley graphs. It also implies that, for example, the rough Cayley graph of $\mathbb{R}$ is the Cayley graph of $\mathbb{Z}$.

Next we shall show that the rough Cayley graph is essentially unique, justifying the terminology used in the previous paragraph. Recall that we consider graphs as discrete metric spaces consisting only of the vertices of the graph. The other natural choice would be to consider graphs as geodesic metric spaces by identifying each edge with the unit interval. However, a quasi-action on the set of vertices may be easily extended to a quasi-action on the whole graph, so the difference is not crucial (the associated constants may grow in the process). Moreover, the proof of Theorem 1 actually works also in the case when we have a quasi-action on the set of vertices, even though the associated metric space is not geodesic. It follows from Theorem 1 that every rough Cayley graph of G is quasi-isometric to G, and hence all rough Cayley graphs of G are mutually quasi-isomorphic. Moreover, the rough Cayley graph describes the coarse geometry of a compactly generated locally compact group. For example, a compactly generated group is *hyperbolic* if it is hyperbolic as a metric space when equipped with the word metric. So a compactly generated group is hyperbolic if and only

if its rough Cayley graph is hyperbolic. Now we show that not only are rough Cayley graphs quasi-isometric but the quasi-actions on them are quasi-conjugate: two quasi-actions $(s,x) \mapsto s \cdot x \colon G \times X \to X$ and $(s,y) \mapsto s \cdot y \colon G \times Y \to Y$ on metric spaces X and Y are said to be *quasi-conjugate* if there is a quasi-isometry $\phi \colon X \to Y$ and a constant $c \geq 0$ such that

$$d_Y(\phi(s \cdot x), s \cdot \phi(x)) \leq c.$$

Theorem 3. *The quasi-actions of a compactly generated group G on two of its rough Cayley graphs are quasi-conjugate.*

Proof. Let X and Y be two generalised Cayley graphs of G. Denote the constants associated with the quasi-actions by C_X, r_X, C_Y, r_Y. Fix $x_0 \in X$ and $y_0 \in Y$. For each $x \in X$, choose $s \in G$ such that $d_X(x, s \cdot x_0) \leq r_X$ and define a function $\phi \colon X \to Y$ by $\phi(x) = s \cdot y_0$. By choice we take $\phi(x_0) = y_0$. It follows from Theorem 1 that there are constants $C_\phi \geq 1$ and $r_\phi \geq 0$ such that

$$C_\phi^{-1} d_X(x_1, x_2) - r_\phi \leq d_Y(\phi(x_1), \phi(x_2)) \leq C_\phi d_X(x_1, x_2) + r_\phi$$

for every $x_1, x_2 \in X$. We skip the routine but lengthy calculation.

Let $s \in G$ and $x \in X$. By definition, $\phi(x) = t \cdot y_0$ for some $t \in G$ such that $d_X(x, t \cdot x_0) \leq r_X$. Then

$$\begin{aligned} &d_Y(\phi(s \cdot x), s \cdot \phi(x)) \\ &\qquad \leq d_Y(\phi(s \cdot x), \phi(s \cdot t \cdot x_0)) + d_Y(\phi(s \cdot t \cdot x_0), s \cdot \phi(x)) \\ &\qquad \leq C_\phi C_X d_X(x, t \cdot x_0) + C_\phi r_X + r_\phi + d_Y(u \cdot y_0, s \cdot t \cdot y_0) \end{aligned}$$

where $u \in G$ is such that $d_X(s \cdot t \cdot x_0, u \cdot x_0) \leq r_X$. Next we shall also need the constants $C_{X,G}$, $r_{X,G}$, $C_{Y,G}$ and $r_{Y,G}$ associated with the quasi-isometries $s \mapsto s \cdot x_0 \colon G \to X$ and $s \mapsto s \cdot y_0 \colon G \to Y$. Continuing the calculation, we have

$$\begin{aligned} &d_Y(\phi(s \cdot x), s \cdot \phi(x)) \\ &\qquad \leq C_\phi C_X r_X + C_\phi r_X + r_\phi + d_Y(u \cdot y_0, st \cdot y_0) + r_Y \\ &\qquad \leq C_\phi r_X (C_X + 1) + r_\phi + C_{Y,G} d_G(u, st) + r_{Y,G} + r_Y \\ &\qquad \leq C_\phi r_X (C_X + 1) + r_\phi + C_{Y,G} C_{X,G} (d_X(u \cdot x_0, st \cdot x_0) + r_{X,G}) \\ &\qquad \quad + r_{Y,G} + r_Y \\ &\qquad \leq C_\phi r_X (C_X + 1) + r_\phi + C_{Y,G} C_{X,G} (2 r_X + r_{X,G}) + r_{Y,G} + r_Y. \end{aligned}$$

Hence the quasi-actions on X and Y are quasi-conjugate. □

Krön and Möller [6] defined the rough Cayley graph of a compactly generated totally disconnected group G as a connected locally finite graph on which G acts transitively in such a way that the stabilisers of the vertices are compact open subgroups of G (local finiteness is actually not part of the definition in [6] but is often used as an additional condition). Such a graph may be realised as the homogeneous space

G/U with respect to a compact open subgroup U. It is not very difficult to show that a rough Cayley graph à la Krön and Möller is a rough Cayley graph in our sense: that is, the action is cobounded, proper and uniformly represented. Then by Theorem 3 any rough Cayley graph in our sense is quasi-conjugate to a rough Cayley graph à la Krön and Möller (and so the quasi-action is in this case quasi-conjugate to an isometric action).

4. Lattices in groups

A class of examples of rough Cayley graphs is obtained by considering lattices in groups. Let G be a compactly generated group. A *cocompact lattice* in G is a discrete subgroup Γ of G such that the homogeneous space $\Gamma \setminus G$ is compact. Note that every discrete subgroup Γ is always uniformly discrete: take a symmetric neighbourhood V of the identity e such that $V^2 \cap \Gamma = \{e\}$, and then $xV \cap yV = \emptyset$ whenever $x \neq y$ are in Γ.

Let Γ be a cocompact lattice in G and let $\pi\colon G \to \Gamma \setminus G$ denote the quotient map. Since $\Gamma \setminus G$ is compact, there is a compact set $K \subseteq G$ such that $\pi(K) = \Gamma \setminus G$. It follows that there is a relatively compact, symmetric, open neighbourhood U of the identity such that U generates G and and $\pi(U) = \Gamma \setminus G$.

The following lemma is essentially proved in [7, section 0.40].

Lemma 4. *There is a finite set $F \subseteq \Gamma$ such that F generates Γ and $\Gamma \cap U^n \subseteq F^n$ for every positive integer n.*

Proof. If $s \in G$, then $su^{-1} \in \Gamma$ for some $u \in U$, and so $G = \Gamma U$. Since U is relatively compact and $G = \Gamma U$, there is a finite set $F \subseteq \Gamma$ such that $U^2 \subseteq FU$ and $\Gamma \cap U \subseteq F$. Inductively, $U^n \subseteq F^{n-1}U$. Now if $x \in \Gamma \cap U^n$ then $x = yu$ for some $y \in F^{n-1}$ and $u \in U$. Hence $u \in \Gamma \cap U \subseteq F$, and so $x \in F^n$. This proves both statements at once. □

Theorem 5. *Suppose that Γ is a cocompact lattice in a compactly generated locally compact group G. Then there is a cobounded, proper, uniformly represented quasi-action of G on the Cayley graph of Γ. That is, the Cayley graph of Γ is the rough Cayley graph of G.*

Proof. We consider the Cayley graph of Γ with respect to a finite generating set F that satisfies Lemma 4 (any other generating set F_1 satisfies $\Gamma \cap U^n \subseteq F_1^{kn}$ for some constant k so there is no loss of generality in the choice of F). We define the quasi-action on Γ similarly as in the proof of Theorem 2: for a given $x \in \Gamma$, choose an open neighbourhood V of the identity in G such that $x^{-1}V^2x \subseteq U$ and continue as in Theorem 2. Then

$$s \cdot x \in sxU^2.$$

Now if $x, y \in \Gamma$ with $d_\Gamma(x, y) = n$. Then $x^{-1}y \in F^n$ and it follows that

$$(s \cdot x)^{-1}(s \cdot y) \in U^2 F^n U^2 \subseteq U^{mn+4}$$

where m is a constant such that $F \subseteq U^m$. By Lemma 4,

$$d_\Gamma(s \cdot x, s \cdot y) \leq mn + 4.$$

Verifying all the remaining properties is similar: follow the proof of Theorem 2 and apply Lemma 4 when needed. □

5. Growth

As a simple illustration that the rough Cayley graph does capture some important properties of locally compact groups, we shall show that compactly generated groups have the same growth as the associated rough Cayley graphs. It is worth noting that in this case the actual Cayley graph is useless.

Let G be a locally compact group generated by a compact symmetric neighbourhood U, and denote the left Haar measure of G by λ. Recall that G has *polynomial growth* if $\lambda(U^m)$ is bounded by a polynomial in m, *intermediate growth* if it does not have polynomial growth but $\limsup \lambda(U^m)^{1/m} = 1$, and *exponential growth* otherwise. The definitions are independent on the choice of U. The growth of a connected locally finite graph is defined similarly, replacing $\lambda(U^m)$ by the cardinality of vertices in $N_m(x_0)$, where x_0 is a fixed base point. Note that the growth does not depend on the chosen base point, although the actual values of $|N_m(x_0)|$ may.

We now fix notation for the rest of the section (some of the notation is actually slightly more complicated than needed in this section but is introduced for the purposes of the next section). Let X be a rough Cayley graph of a compactly generated locally compact group G. Fix a base point $x_0 \in X$. Since the quasi-action is uniformly represented, there is $T \subseteq G$ such that $G \cdot x_0 = T \cdot x_0$ and T is right uniformly discrete with respect to some compact neighbourhood V of the identity. Let $r \geq 0$ such that $N_r(G \cdot x_0) = X$. For every $A \subseteq X$, define

$$\widetilde{A} = \{\, t \in T;\ N_r(t \cdot x_0) \cap A \neq \emptyset \,\}.$$

Let $K \subseteq G$ be a compact neighbourhood of the identity such that K generates G; we may suppose that $V \subseteq K$. We consider G a metric space with the word metric defined with respect to K. Let C and r be the constants associated with the quasi-isometry $s \mapsto s \cdot x_0$ (again we combine two r's):

$$C^{-1} d_G(s, t) - r \leq d(s \cdot x_0, t \cdot x_0) \leq C d_G(s, t) + r.$$

To simplify notation we assume that C and r are integers.

The following lemma collects some nuggets of information needed for moving between G and X.

Lemma 6. *Let $A \subseteq X$.*

(i) *$\widetilde{A}$ is right uniformly discrete with respect to V.*
(ii) *$M^{-r}|A| \leq |\widetilde{A}| \leq M^r|A|$ where M is a uniform bound on the degree of vertices in X.*
(iii) *If $s \cdot x_0 \in A$, then $s \in \widetilde{A}K^{Cr}$.*
(iv) *If $s \in \widetilde{A}K^m$, then $d(s \cdot x_0, A) \leq Cm + 2r$.*

Proof. The first statement is immediate from the definition of $\widetilde{A}$.

The second statement follows from the containments $\widetilde{A} \cdot x_0 \subseteq N_r(A)$ and $A \subseteq N_r(\widetilde{A} \cdot x_0)$.

The third statement holds because $s \cdot x_0 = t \cdot x_0$ for some $t \in \widetilde{A}$ and hence $d_G(s,t) \leq Cd(s \cdot x_0, t \cdot x_0) + Cr = Cr$.

The fourth statement is kind of a converse of the third: $d_G(s,t) \leq m$ for some $t \in \widetilde{A}$, so $d(s \cdot x_0, A) \leq Cm + 2r$. □

Theorem 7. *A compactly generated locally compact group and its rough Cayley graph have the same growth.*

Proof. Let m be a positive integer, and suppose that $s \in K^m$. Then $d(s \cdot x_0, x_0) \leq Cm + r$ and $s \in N_{Cm+r}(x_0)^{\sim}K^{Cr}$ by Lemma 6. Therefore $K^m \subseteq N_{Cm+r}(x_0)^{\sim}K^{Cr}$ and so

$$\lambda(K^m) \leq |N_{Cm+r}(x_0)^{\sim}|\lambda(K^{Cr}) \leq M^r\lambda(K^{Cr})|N_{Cm+r}(x_0)|. \tag{3}$$

Conversely, if $s \in N_m(x_0)^{\sim}$, then $d(s \cdot x_0, x_0) \leq m + r$ and so $d_G(s, e) \leq C(m + 2r)$. It follows that

$$N_m(x_0)^{\sim}V \subseteq K^{C(m+2r)+1},$$

and hence

$$|N_m(x_0)| \leq \frac{M^r}{\lambda(V)}\lambda(K^{C(m+2r)+1}). \tag{4}$$

Combining (3) and (4) we see that X and G have the same growth. □

6. Amenability

We follow the terminology of [2]. A *quasi-lattice* in a metric space X is a subset $\Gamma \subseteq X$ such that Γ is coarsely dense (i.e. there is $c > 0$ such that $N_c(\Gamma) = X$) and for every $c > 0$ there is $R > 0$ such that

$$|\Gamma \cap N_c(x)| < R$$

for every $x \in X$. A metric space is said to be of *bounded geometry* if there exists a quasi-lattice in X. For example, the vertex set of a uniformly locally finite graph is a quasi-lattice in the geodesic metric space determined by the graph, and a maximal right uniformly discrete subset of a compactly generated group is a quasi-lattice with respect to the word metric.

In a metric space X, the *c-boundary* of a set $A \subseteq X$ is

$$\partial_c A = \{ x \in X;\ d(x, A) \le c \text{ and } d(x, X \setminus A) \le c \}.$$

Consider a metric space X with bounded geometry, and let Γ be a quasi-lattice in X. Then X is *amenable* if for every $c > 0$ and $\epsilon > 0$ there is a finite set $A \subseteq \Gamma$ such that

$$\frac{|\partial_c A|}{|A|} < \epsilon,$$

where the boundary of A is calculated inside Γ. It is known [2] that amenability of metric spaces with bounded geometry is invariant under quasi-isometries (and so does not depend on the choice of the quasi-lattice). Therefore it follows from Theorem 1 that the rough Cayley graph of a compactly generated group G is amenable if and only if G is amenable *as a metric space* with respect to the word metric. Hence the main result of this section implies that a unimodular G is amenable as a metric space if and only if it is amenable.

Recall that a locally compact group G is amenable if and only if for every $\epsilon > 0$ and for every compact $F \subseteq G$ containing the identity there is a non-null compact set $L \subseteq G$ such that

$$\frac{\lambda(FL \setminus L)}{\lambda(L)} < \epsilon.$$

(This slightly unusual formulation of amenability is due to Emerson and Greenleaf [5]; see also [9].) It turns out that with our choice of notation, the right-handed version of the above definition is actually the right one: replace FL with LF and the left Haar measure λ with the right Haar measure. However, we shall only consider the unimodular case, so the latter part of the remark may be ignored: λ is both left and right invariant.

Theorem 8. *Suppose that G is a unimodular compactly generated group. The rough Cayley graph of G is amenable if and only if G is amenable.*

Proof. We continue with the notation set up before Lemma 6. Suppose first that the rough Cayley graph X is amenable. Since every compact set is contained in some K^m, it is enough to deal with these sets. Given an integer m and $\epsilon > 0$, choose $A \subseteq X$ such that

$$\frac{|\partial_{C(m+1)+(C^2+2)r} A|}{|A|} < \epsilon.$$

Define $L = \widetilde{A} K^{Cr+1}$.

Claim:

$$LK^m \setminus L \subseteq (\partial_{C(m+1)+(C^2+2)r} A)^{\sim} K^{Cr}.$$

Let s be in the left-hand side set. If $s \cdot x_0 \in A$, then $s \in L$ by Lemma 6. On the other hand, $s \in \widetilde{A}K^{m+1+Cr}$ so

$$d(s \cdot x_0, A) \leq C(m+1) + (C^2+2)r.$$

Hence $s \cdot x_0 \in \partial_{C(m+1)+(C^2+2)r}A$ and the claim follows by Lemma 6 again.

It follows from the claim that

$$\lambda(LK^m \setminus L) \leq M^r \lambda(K^{Cr}) |\partial_{C(m+1)+(C^2+2)r}A|.$$

On the other hand, $L \supseteq \widetilde{A}V$ and $\widetilde{A}$ is right uniformly discrete with respect to V, so $\lambda(L) \geq |\widetilde{A}|\lambda(V)$. Therefore

$$\frac{\lambda(LK^m \setminus L)}{\lambda(L)} \leq \frac{M^{2r}\lambda(K^{Cr})}{\lambda(V)} \frac{|\partial_{C(m+1)+(C^2+2)r}A|}{|A|} \leq \frac{M^{2r}\lambda(K^{Cr})}{\lambda(V)}\epsilon.$$

Consequently, G is amenable.

Conversely, suppose that G is amenable. Let $\epsilon > 0$ and let m be a positive integer. Write $F = K^{C(2m+2r+1)+r}V$. Then there is a compact set $L \subseteq G$ such that

$$\frac{\lambda(LF \setminus L)}{\lambda(L)} < \epsilon.$$

Define

$$A = N_{m+r+1}(L \cdot x_0).$$

Claim 2:

$$(\partial_m A)^{\sim} V \subseteq LF \setminus L.$$

Let $s \in (\partial_m A)^{\sim}$. Then there is $x \in \partial_m A$ such that $d(s \cdot x_0, x) \leq r$ and hence $t \in L$ such that $d(s \cdot x_0, t \cdot x_0) \leq 2m + 2r + 1$. Therefore $s \in LK^{C(2m+2r+1)+r}$. If $sv \in L$ for some $v \in V$, then

$$d(sv \cdot x_0, x) = d(s \cdot x_0, x) \leq r$$

implies that $d(L \cdot x_0, X \setminus A) \leq m + r$. This is a contradiction so $sv \notin L$ and claim 2 is proved.

It follows from claim 2 that

$$|\partial_m A| \leq \frac{M^r}{\lambda(V)}\lambda(LF \setminus L). \tag{5}$$

On the other hand, if $s \in L$, then $s \cdot x_0 \in A$ and hence $s \in \widetilde{A}K^{Cr}$. Therefore

$$\lambda(L) \leq |\widetilde{A}|\lambda(K^{Cr})$$

and so

$$|A| \geq \frac{M^{-r}}{\lambda(K^{Cr})}\lambda(L). \tag{6}$$

Combining the approximations (5) and (6) we have

$$\frac{|\partial_m A|}{|A|} \leq \frac{M^{2r}\lambda(K^{Cr})}{\lambda(V)} \frac{\lambda(LF \setminus L)}{\lambda(L)} \leq \frac{M^{2r}\lambda(K^{Cr})}{\lambda(V)}\epsilon.$$

Consequently, X is amenable. □

Corollary 9. *Suppose that G_1 and G_2 are unimodular compactly generated locally compact groups that are quasi-isometric. Then G_1 is amenable if and only if G_2 is amenable.*

Combining Theorems 8 and 5, we get the following.

Corollary 10. *Let G be a unimodular compactly generated locally compact group and Γ a discrete subgroup of G such that the homogeneous space $\Gamma \setminus G$ is compact (i.e. Γ is a cocompact lattice). Then G is amenable if and only if Γ is amenable.*

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